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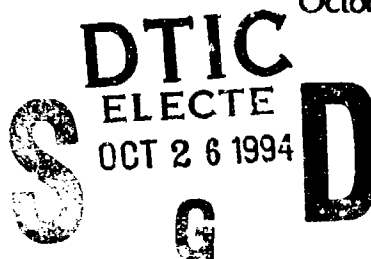


Nonparametric Estimation of the Cyclic Cross-Spectrum

by Brian Sadler

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13. ABSTRACT (Maximum 200 words) Cyclostationary processes are an important class of nonstationary processes. In this report we consider nonparametric estimation of the cyclic cross-spectrum. A periodogram-based estimator is studied and its asymptotic behavior characterized. This extends the recent univariate work of Dandawate and Giannakis to the multivariate case. The results are useful for a variety of multi-sensor cyclostationary signal processing scenarios, such as bearing estimation.				
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1. Introduction

In this report we study the statistics of a periodogram-based estimate of the cyclic cross-spectrum. Our multivariate work is based heavily upon the univariate work of Dandawate and Giannakis [1,2], in which they developed the theory of periodogram-based estimates of the cyclic spectrum of a single cyclostationary process. Their work extended conventional spectrum estimation theory for stationary time series to the cyclostationary case, especially relying on the exposition of Brillinger [3] and Brillinger and Rosenblatt [4,5]. We simply extend the work of Dandawate and Giannakis to the multivariate case. The present work can thus be viewed as a generalization of the multivariate cross-periodogram spectrum estimation theory for stationary processes as put forward by Brillinger [3].

The primary motivation for extending to the multivariate case is to handle multiple time series problems, as might arise in multi-sensor signal processing scenarios. Because of the dimensionality difficulties involved we focus on the second-order cross-spectrum case and do not consider estimates of higher order cyclic cross-cumulants or polyspectra, while noting that the univariate cyclostationary theory has been extended to include arbitrary k th-order cyclic statistics [1].

Cyclic (cross)-spectrum estimates based on the periodogram are particularly appealing due to their nonparametric nature, and because they allow the use of the fast Fourier transform (FFT) algorithm, which speeds computation. It is shown that such estimates of the cyclic cross-spectrum are consistent and asymptotically normally distributed under mild conditions on the time series. This will allow for optimal criteria to be developed for detection and estimation schemes based on the cyclic cross-spectrum. For example, the optimal estimation of the time difference of arrival between two sensors is key for bearing estimation schemes, and this delay can be estimated for cyclostationary signals via the cyclic cross-spectrum. Thus, the results of this report will prove useful in the analysis of such multi-sensor scenarios.

2. Background

Consider the vector-valued time series $\mathbf{X}(t)$, for $t = 0, 1, 2, \dots$, with real-valued components $x_i(t)$, $i = 1, \dots, r$. These might arise as the outputs of multiple sensors, for example. We assume without loss of generality that $E[x_i(t)] = 0$ for each choice of i . With $x_i(t)$ and $x_j(t)$ components of $\mathbf{X}(t)$, we define their *time-varying cross-covariance* to be

$$c_{ij}(t; \tau) \triangleq E[x_i(t + \tau)x_j(t)]. \quad (1)$$

If $x_i(t)$ and $x_j(t)$ are stationary, then $c_{ij}(t; \tau) = c_{ij}(\tau)$ is the conventional cross-covariance and exhibits no time dependence. Background on the stationary cross-covariance can be found, for example, in Priestly [6, chapt. 9]. If $x_i(t)$ and $x_j(t)$ are cyclostationary, then $c_{ij}(t; \tau)$ is periodic (or almost periodic) in t , so for each fixed lag τ the time-varying covariance can be expanded in a Fourier series (e.g., see Besicovitch [7]). The resulting Fourier series coefficients are the *cyclic cross-covariance*, given by

$$C_{ij}(\alpha; \tau) \triangleq \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} c_{ij}(t; \tau) e^{-j\alpha t}, \quad (2)$$

with α called the *cycle frequency*. In analogy with the stationary case, we may also consider the Fourier transform of $c_{ij}(t; \tau)$ and $C_{ij}(\alpha; \tau)$ with respect to τ , yielding, respectively, the *time-varying spectrum*

$$s_{ij}(t; \omega) \triangleq \sum_{\tau=-\infty}^{\infty} c_{ij}(t; \tau) e^{-j\omega\tau}, \quad (3)$$

and the *cyclic spectrum*, given by

$$S_{ij}(\alpha; \omega) \triangleq \sum_{\tau=-\infty}^{\infty} C_{ij}(\alpha; \tau) e^{-j\omega\tau}. \quad (4)$$

To obtain the "auto" versions of these functions we simply set $i = j$ so that, for example, $c_{ii}(t; \tau) = c_i(t; \tau)$ is the time-varying autocorrelation.

Note that $C_{ij}(0; \tau)$ is the correlation as conventionally defined for stationary processes. Similarly, $S_{ij}(0; \omega)$ is the power spectrum, defined as the Fourier transform of $C_{ij}(0; \tau)$. Thus, the statistics of stationary processes are a special case of cyclostationary statistics with cycle frequency $\alpha = 0$, which corresponds to periodic statistics with period zero. Note also that $C_{ij}(\alpha; \tau)$ and $S_{ij}(\alpha; \omega)$ are time-invariant quantities, which is a fundamental reason why cyclic statistics are an important tool for analysis of nonstationary processes.

In this report we analyze the properties of a periodogram-based estimate of $S_{ij}(\alpha; \omega)$. We make use of the *finite Fourier transform* of $x_i(t)$, defined as

$$X_i^{(T)}(\omega) \triangleq \sum_{t=0}^{T-1} x_i(t) e^{-j\omega t}, \quad (5)$$

and the *cyclic cross-periodogram*, defined as

$$I_{ij}^{(T)}(\alpha; \omega) \triangleq \frac{1}{T} X_i^{(T)}(\omega) \{X_j^{(T)}(\omega - \alpha)\}^*, \quad (6)$$

where * denotes complex conjugate. Since $x_i(t)$ is real valued, this may be written

$$I_{ij}^{(T)}(\alpha; \omega) = \frac{1}{T} X_i^{(T)}(\omega) X_j^{(T)}(\alpha - \omega). \quad (7)$$

The definition of equation (6) is a generalization of the univariate version of the definition used by Hurd [8].

In deriving the covariance of $I_{ij}^{(T)}(\alpha; \omega)$, we make use of higher order cumulants. All of the above definitions may be generalized to higher order cumulant cases. The "auto" higher order versions are defined by Dandawate [1]. Under the assumption that $E[|x_i(t)|^k] < \infty$, we define the *joint cumulant of order k* of $\mathbf{X}(t)$ as,[†]

$$c_{a_1, \dots, a_k}(t_1, \dots, t_k) \triangleq \text{cum}\{x_{a_1}(t_1), \dots, x_{a_k}(t_k)\}, \quad (8)$$

for $k \geq 2$. In the case of stationary processes

$$c_{a_1, \dots, a_k}(t_1, \dots, t_k) = c_{a_1, \dots, a_k}(\tau_1, \dots, \tau_{k-1}), \quad (9)$$

while in the time-varying case, in analogy with equation (1), we explicitly maintain the time dependence as

$$c_{a_1, \dots, a_k}(t_1, \dots, t_k) = c_{a_1, \dots, a_k}(t; \tau_1, \dots, \tau_{k-1}). \quad (10)$$

For $\mathbf{X}(t)$ cyclostationary then, in extension of equation (2), we define,

$$C_{a_1, \dots, a_k}(\alpha; \tau_1, \dots, \tau_{k-1}) \triangleq \lim_{T \rightarrow \infty} \sum_{t=0}^{T-1} c_{a_1, \dots, a_k}(t; \tau_1, \dots, \tau_{k-1}) e^{-j\alpha t}, \quad (11)$$

to be the *kth-order cyclic cumulant* of $\mathbf{X}(t)$ at cycle frequency α . The time-varying cumulant spectrum (polyspectrum) is written

$$s_{a_1, \dots, a_k}(t; \omega_1, \dots, \omega_{k-1}) \triangleq \sum_{\tau_1=-\infty}^{\infty} \dots \sum_{\tau_{k-1}=-\infty}^{\infty} c_{a_1, \dots, a_k}(t; \tau_1, \dots, \tau_{k-1}) e^{-j(\omega_1 \tau_1 + \dots + \omega_{k-1} \tau_{k-1})}, \quad (12)$$

and the *cyclic cumulant spectrum* is defined as

$$S_{a_1, \dots, a_k}(\alpha; \omega_1, \dots, \omega_{k-1}) \triangleq \sum_{\tau_1=-\infty}^{\infty} \dots \sum_{\tau_{k-1}=-\infty}^{\infty} C_{a_1, \dots, a_k}(\alpha; \tau_1, \dots, \tau_{k-1}) e^{-j(\omega_1 \tau_1 + \dots + \omega_{k-1} \tau_{k-1})}. \quad (13)$$

[†]See Brillinger [3, p. 21]. We use the notation $c_{a_1, \dots, a_k}(\cdot)$ to denote arbitrary higher order cumulants of k th order, rather than using $c_{y,k}(\cdot)$. For fixed order we retain the previous notation, so, for example, we use $c_{ij}(\cdot)$ rather than $c_{a_1, a_2}(\cdot)$ in the second-order case.

3. Cyclic Cross-Periodogram

In this section we explore the cyclic cross-periodogram $I_{ij}^{(T)}(\alpha; \omega)$, defined by equation (6), as a potential estimate of the cyclic cross-spectrum $S_{ij}(\alpha; \omega)$. The asymptotic properties of $I_{ij}^{(T)}(\alpha; \omega)$ are summarized in a theorem, which shows that $I_{ij}^{(T)}(\alpha; \omega)$ does not result in consistent estimates of $S_{ij}(\alpha; \omega)$. This is analogous to the stationary case, where the conventional cross-periodogram results in inconsistent estimates of the power spectrum, e.g., see Priestly [6, chapt. 9]. As is well known in the stationary case, the key to obtaining consistent estimates is smoothing the periodogram. As shown by Dandawate and Giannakis, this idea extends to the cyclic case as well [1,2].

To begin our study of $I_{ij}^{(T)}(\alpha; \omega)$, we state assumptions on the time series $\mathbf{X}(t)$. These mixing conditions insure that samples of $\mathbf{X}(t)$ that are well separated in time are becoming statistically independent. For a general discussion of mixing conditions, see Brillinger [3, chapt. 1].

Assumption 1a

$$\sum_{\tau_1=-\infty}^{\infty} \cdots \sum_{\tau_k=-\infty}^{\infty} |\tau_k| |c_{a_1, \dots, a_{k-1}}(t; \tau_1, \dots, \tau_{k-1})| < \infty, \text{ for } i = 1, \dots, k-1, \forall t, \forall k. \quad (14)$$

Assumption 1b

$$\sum_{\tau=-\infty}^{\infty} |\tau| |c_{ij}(t; \tau)| < \infty, \forall t. \quad (15)$$

Assumption 2a

$$\sum_{\tau_1=-\infty}^{\infty} \cdots \sum_{\tau_k=-\infty}^{\infty} |\tau_k| |C_{a_1, \dots, a_{k-1}}(\alpha; \tau_1, \dots, \tau_{k-1})| < \infty, \text{ for } i = 1, \dots, k-1, \forall \alpha, \forall k. \quad (16)$$

Assumption 2b

$$\sum_{\tau} |\tau| |C_{ij}(\alpha; \tau)| < \infty, \forall \alpha. \quad (17)$$

Note that Assumptions 1b and 2b are the second-order cases ($k = 2$) of the more general Assumptions 1a and 2a, respectively. For clarity we have included Assumptions 1b and 2b so we can refer specifically to the second-order case when desired. We proceed by stating a lemma that characterizes the rate of convergence (as $T \rightarrow \infty$) of the cyclic cumulant spectrum. In the following the notation $a_n = O(b_n)$ means that $|a_n/b_n|$ is bounded for n sufficiently large, e.g., see Brillinger [3, chapt. 2].

Lemma 1 *If Assumption 1a holds, then*

$$\left| S_{a_1, \dots, a_k}(t; \omega_1, \dots, \omega_{k-1}) - \sum_{\tau_1 = -(T-1)}^{(T-1)} \dots \sum_{\tau_{k-1} = -(T-1)}^{(T-1)} c_{a_1, \dots, a_k}(t; \tau_1, \dots, \tau_{k-1}) e^{-j(\omega_1 \tau_1 + \dots + \omega_{k-1} \tau_{k-1})} \right| = \mathcal{O}(T^{-1}). \quad \square \quad (18)$$

For the case of $k = 2$, Lemma 1 reduces to the following corollary.

Corollary 1 *If Assumption 1b holds, then*

$$\left| s_{ij}(t; \omega) - \sum_{\tau = -(T-1)}^{(T-1)} c_{ij}(t; \tau) e^{-j\omega \tau} \right| = \mathcal{O}(T^{-1}). \quad \square \quad (19)$$

The next lemma describes the rate of convergence of the finite Fourier transform of the k th-order time-varying cumulant spectrum. We are generalizing Theorem 4.3.2 of Brillinger [3]. The proof is similar to that used by Dandawate, which is a generalization of the stationary case of Brillinger and Rosenblatt [4].

Lemma 2 *If assumption 2a holds, then*

$$\text{cum}\{X_{a_1}^{(T)}(\omega_1), \dots, X_{a_k}^{(T)}(\omega_k)\} = \sum_{t=0}^{T-1} S_{a_1, \dots, a_k}(t; \omega_1, \omega_2, \dots, \omega_{k-1}) e^{-j(\omega_1 + \dots + \omega_k)t} + \mathcal{O}(1). \quad \square \quad (20)$$

For the case of $k = 2$, lemma 2 reduces to the following corollary.

Corollary 2 *If assumption 2b holds then,*

$$\text{cum}\{X_i^{(T)}(\omega_0), X_j^{(T)}(\omega_1)\} = \sum_{t=0}^{T-1} S_{ij}(t; \omega_0) e^{-j(\omega_0 + \omega_1)t} + \mathcal{O}(1). \quad \square \quad (21)$$

We are now prepared to state the main result of this section in the following theorem that describes the asymptotic bias and variance of the cyclic cross-periodogram $I_{ij}^{(T)}(\alpha; \omega)$. The proof of the theorem relies on the above lemmas.

Theorem 1 (Bias and variance of the cyclic cross-periodogram).

Suppose $\mathbf{X}(t)$ is cyclostationary with zero mean as defined in section 2. If $S_{ijkl}(\alpha; \omega_1, \omega_2, \omega_3)$ exists and is finite, and Assumptions 1 and 2 hold, then

$$S_{ij}(\alpha; \omega) = \lim_{T \rightarrow \infty} E[I_{ij}^{(T)}(\alpha; \omega)], \quad (22)$$

and

$$\begin{aligned} \lim_{T \rightarrow \infty} \text{cov}\{I_{ij}^{(T)}(\alpha; \omega), I_{kl}^{(T)}(\beta; \mu)\} &= S_{ik}(\omega - \mu; \omega) S_{jl}(\alpha - \omega + \mu - \beta; \alpha - \omega) \\ &+ S_{il}(\omega + \mu - \beta; \omega) S_{jk}(\alpha - \omega - \mu; \alpha - \omega). \quad \square \end{aligned} \quad (23)$$

It is apparent from equation (22) that $I_{ij}^{(T)}(\alpha; \omega)$ is asymptotically unbiased. However, from equation (23) we conclude that, in general, $I_{ij}^{(T)}(\alpha; \omega)$ has non-zero covariance and is therefore an inconsistent estimator of $S_{ij}(\alpha; \omega)$. A consistent estimator will subsequently be obtained based on a smoothed version of $I_{ij}^{(T)}(\alpha; \omega)$. Before we do this we consider some special cases of equation (23) in the next section, and in particular we develop the asymptotic variance-covariance matrix for $I_{ij}^{(T)}(\alpha; \omega)$.

4. Special Cases and the Variance-Covariance Matrix

In this section we consider some special cases of the results of Theorem 1. First, we give two useful symmetry properties. As we have shown in theorem 1,

$$S_{ij}(\alpha; \omega) = \lim_{T \rightarrow \infty} E[I_{ij}^{(T)}(\alpha; \omega)], \quad (24)$$

so that

$$S_{ij}(\alpha; \omega) = \lim_{T \rightarrow \infty} \frac{1}{T} E[X_i^{(T)}(\omega) X_j^{(T)}(\alpha - \omega)]. \quad (25)$$

From equation (25), and using the fact that the $x_i(t)$ are real-valued, we can deduce the following symmetry properties:

$$S_{ij}(\alpha; \alpha - \omega) = S_{ji}(\alpha; \omega), \quad (26)$$

$$S_{ij}^*(\alpha; \omega) = S_{ij}(-\alpha; -\omega). \quad (27)$$

Next we consider some important cases of equation (23). By setting $i = j = k = l$ and using equation (26) in equation (23), we obtain the univariate results of Dandawate and Giannakis (compare with Dandawate [1,2]),

$$\begin{aligned} \lim_{T \rightarrow \infty} \text{cov}\{I_{ii}^{(T)}(\alpha; \omega), I_{ii}^{(T)}(\beta; \mu)\} &= S_{ii}(\omega - \mu; -\mu) S_{ii}(\alpha - \omega + \mu - \beta; \mu - \beta) \\ &+ S_{ii}(\omega + \mu - \beta; \mu - \beta) S_{ii}(\alpha - \omega - \mu; -\mu). \end{aligned} \quad (28)$$

Another important case that may be obtained from equation (23) is the covariance of the conventional cross-periodogram for stationary processes. From equation (23), we have

$$\lim_{T \rightarrow \infty} \text{cov}\{I_{ij}^{(T)}(0; \omega), I_{kl}^{(T)}(0; \omega)\} = S_{ik}(0; \omega) S_{jl}(0; -\omega) + S_{il}(2\omega; \omega) S_{jk}(-2\omega; -\omega). \quad (29)$$

Note that, for stationary processes, $S_{ij}(\alpha; \omega) \equiv 0$ for $\alpha \neq 0$. In this case equation (29) reduces to

$$\lim_{T \rightarrow \infty} \text{cov}\{I_{ij}^{(T)}(0; \omega), I_{kl}^{(T)}(0; \omega)\} = \begin{cases} S_{ik}(0; \omega)S_{jl}^*(0; \omega), & \omega \neq 0, \pm\pi \\ S_{ik}(0; \omega)S_{jl}(0; \omega) + S_{il}(0; \omega)S_{jk}^*(0; \omega), & \omega = 0, \pm\pi. \end{cases} \quad (30)$$

This expression can be found, for example, in Priestly [6, chapt. 9].

For *periodically correlated* (PC) processes we are restricted to the set of cycle frequencies given by $\alpha \in \{2\pi k/T_0\}$, for $k = 0, 1, \dots, T_0 - 1$. Using (23), this leads to

$$\lim_{T \rightarrow \infty} \text{cov}\{I_{ij}^{(T)}(\frac{2\pi k}{T_0}; \omega), I_{kl}^{(T)}(\frac{2\pi n}{T_0}; \omega)\} = \begin{cases} S_{ik}(0; \omega)S_{jl}(0; \alpha - \omega), & \omega \neq \frac{n\pi}{T_0} \\ S_{ik}(0; \omega)S_{jl}(0; \alpha - \omega) + S_{il}(\frac{2\pi(n-k)}{T_0}; \omega)S_{jk}(\frac{2\pi(k-n)}{T_0}; \frac{2\pi k}{T_0} - \omega), & \omega = \frac{n\pi}{T_0}, \end{cases} \quad (31)$$

for n integer.

To complete our results on the covariance of $I_{ij}^{(T)}(\alpha; \omega)$, we derive its variance-covariance matrix. From equation (23) it follows easily that

$$\lim_{T \rightarrow \infty} \text{cov}\{I_{ij}^{(T)}(\alpha; \omega), I_{kl}^{(T)}(\alpha; \omega)\} = S_{ik}(0; \omega)S_{jl}(0; \alpha - \omega) + S_{il}(2\omega - \alpha; \omega)S_{jk}(\alpha - 2\omega; \alpha - \omega). \quad (32)$$

We now consider some cases of equation (32).

Setting $i = k$ and $j = l$ and using the symmetry properties given by equations (26) and (27), we obtain the following expression for the asymptotic variance of the cyclic cross-periodogram,

$$\text{var}\{I_{ij}^{(T)}(\alpha; \omega)\} \approx S_{ii}(0; \omega)S_{jj}(0; \alpha - \omega) + |S_{ij}(2\omega - \alpha; \omega)|^2. \quad (33)$$

Setting $i = j$ and $k = l$ yields the covariance between two cyclic auto-periodograms,

$$\text{cov}\{I_{ii}^{(T)}(\alpha; \omega), I_{jj}^{(T)}(\alpha; \omega)\} \approx S_{ij}(0; \omega)S_{ij}(0; \alpha - \omega) + S_{ij}(2\omega - \alpha; \omega)S_{ji}^*(2\omega - \alpha; \omega). \quad (34)$$

With $i = j = k = l$ we obtain the variance of a cyclic auto-periodogram,

$$\text{var}\{I_{ii}^{(T)}(\alpha; \omega)\} \approx S_{ii}(0; \omega)S_{ii}(0; \alpha - \omega) + |S_{ii}(2\omega - \alpha; \omega)|^2. \quad (35)$$

We summarize these results in the asymptotic variance-covariance matrix as follows.

	$I_{11}^{(T)}(\alpha; \omega)$	$I_{12}^{(T)}(\alpha; \omega)$	$I_{21}^{(T)}(\alpha; \omega)$	$I_{22}^{(T)}(\alpha; \omega)$
$I_{11}^{(T)}(\alpha; \omega)$	$S_{11}(0; \omega)S_{11}(0; \alpha - \omega) + S_{11}(2\omega - \alpha; \omega) ^2$	$S_{12}(0; \omega)S_{12}(0; \alpha - \omega) + S_{11}(2\omega - \alpha; \omega)S_{11}^*(2\omega - \alpha; \omega)$	$S_{11}(0; \omega)S_{12}(0; \alpha - \omega) + S_{12}(2\omega - \alpha; \omega)S_{11}(2\omega - \alpha; \omega)$	$S_{12}(0; \omega)S_{11}(0; \alpha - \omega) + S_{11}(2\omega - \alpha; \omega)S_{12}^*(2\omega - \alpha; \omega)$
$I_{12}^{(T)}(\alpha; \omega)$	$S_{12}(0; \omega)S_{11}(0; \alpha - \omega) + S_{11}(2\omega - \alpha; \omega)S_{12}^*(2\omega - \alpha; \omega)$	$S_{12}(0; \omega)S_{12}(0; \alpha - \omega) + S_{12}(2\omega - \alpha; \omega) ^2$	$S_{11}(0; \omega)S_{12}(0; \alpha - \omega) + S_{12}(2\omega - \alpha; \omega)S_{11}^*(2\omega - \alpha; \omega)$	$S_{12}(0; \omega)S_{11}(0; \alpha - \omega) + S_{11}(2\omega - \alpha; \omega)S_{12}^*(2\omega - \alpha; \omega)$
$I_{21}^{(T)}(\alpha; \omega)$	$S_{11}(0; \omega)S_{12}(0; \alpha - \omega) + S_{11}(2\omega - \alpha; \omega)S_{12}^*(2\omega - \alpha; \omega)$	$S_{12}(0; \omega)S_{12}(0; \alpha - \omega) + S_{12}(2\omega - \alpha; \omega)S_{12}^*(2\omega - \alpha; \omega)$	$S_{11}(0; \omega)S_{12}(0; \alpha - \omega) + S_{12}(2\omega - \alpha; \omega) ^2$	$S_{12}(0; \omega)S_{11}(0; \alpha - \omega) + S_{11}(2\omega - \alpha; \omega)S_{12}^*(2\omega - \alpha; \omega)$
$I_{22}^{(T)}(\alpha; \omega)$	$S_{12}(0; \omega)S_{11}(0; \alpha - \omega) + S_{11}(2\omega - \alpha; \omega)S_{12}^*(2\omega - \alpha; \omega)$	$S_{12}(0; \omega)S_{12}(0; \alpha - \omega) + S_{12}(2\omega - \alpha; \omega)S_{12}^*(2\omega - \alpha; \omega)$	$S_{12}(0; \omega)S_{12}(0; \alpha - \omega) + S_{12}(2\omega - \alpha; \omega)S_{11}^*(2\omega - \alpha; \omega)$	$S_{12}(0; \omega)S_{11}(0; \alpha - \omega) + S_{11}(2\omega - \alpha; \omega) ^2$

(36)

5. Smoothed Cyclic Cross-Periodogram

It is well known, for the case of stationary processes, that the periodogram leads to inconsistent estimates. Consistent estimates are obtained by convolving the periodogram with a spectral window, which amounts to performing local averaging. The choice of window allows the user to perform a bias/variance tradeoff in the resulting estimate. In our case, as we have seen in Theorem 1, the cyclic cross-periodogram $I_{ij}^{(T)}(\alpha; \omega)$ is an inconsistent estimator of the cyclic cross-spectrum $S_{ij}(\alpha; \omega)$. As shown by Dandawate and Giannakis [1,2], smoothing of the cyclic periodogram results in consistent estimates of the cyclic (auto)-spectrum. Here we extend the idea to the cyclic cross-spectrum with the same results.

Consider the smoothed cyclic cross-periodogram estimate of the cyclic cross-spectrum given by

$$\hat{S}_{ij}^{(T)}(\alpha; \omega) \triangleq \frac{1}{T} \sum_{s=0}^{T-1} I_{ij}^{(T)}(\alpha; \frac{2\pi s}{T}) W^{(T)}(\omega - \frac{2\pi s}{T}). \quad (37)$$

The spectral window $W^{(T)}(\omega)$ obeys certain well known properties, e.g., see Brillinger [3, chapt. 5]. These are briefly outlined next in Assumption 3.

Assumption 3

$W(\omega)$ is an even, real-valued function satisfying

$$\int_{-\infty}^{\infty} W(\omega) d\omega = 1, \quad \int_{-\infty}^{\infty} |W(\omega)| d\omega < \infty. \quad (38)$$

The spectral window weighting function $W^{(T)}(\omega)$ is constructed from $W(\omega)$ as

$$W^{(T)}(\omega) = \sum_{n=-\infty}^{\infty} \frac{1}{B_T} W(B_T^{-1}(\omega + 2\pi n)), \quad (39)$$

where B_T is a sequence for $T = 1, 2, \dots$, with $B_T > 0$, $B_T \rightarrow 0$, and $B_T T \rightarrow \infty$ as $T \rightarrow \infty$.

We now present a theorem that describes the asymptotic behavior of $\hat{S}_{ij}^{(T)}(\alpha; \omega)$, given by equation (37), demonstrating its consistency and asymptotic normality.

Theorem 2

Suppose $\mathbf{X}(t)$ is cyclostationary with zero mean as defined in section 2. Under Assumptions 1 through 3, then

$$\lim_{T \rightarrow \infty} \hat{S}_{ij}^{(T)}(\alpha; \omega) = S_{ij}(\alpha; \omega), \quad (40)$$

and

$$\lim_{T \rightarrow \infty} B_T T \text{cov}\{\hat{S}_{ij}^{(T)}(\alpha; \omega), \hat{S}_{jk}^{(T)}(\alpha; \mu)\} = \lim_{T \rightarrow \infty} \mathcal{E}_w \text{cov}\{I_{ij}^{(T)}(\alpha; \omega), I_{jk}^{(T)}(\alpha; \mu)\}, \quad (41)$$

where $\mathcal{E}_w = \int_{-\infty}^{\infty} |W(\tau)|^2 d\tau$ is the window energy and the cyclic cross-periodogram covariance is given in Theorem 1. Also, the resulting cyclic spectrum estimate is asymptotically normally distributed.

6. Conclusions

Theorem 2 is the main result of this report and is useful for a number of reasons. The finite Fourier transform $X^{(T)}(\omega)$, as defined by equation (5), can be evaluated using the FFT algorithm for composite record length T , so $I_{ij}^{(T)}(\alpha; \omega)$ can be readily obtained. The smoothing step necessary to obtain $\hat{S}_{ij}^{(T)}(\alpha; \omega)$ via equation (37) is a straightforward local averaging procedure. Thus, the smoothed cyclic cross-periodogram estimate of the cyclic cross-spectrum can be readily computed.

The asymptotic variance expression of theorem 1, combined with the asymptotic normality of the estimate, allows for asymptotically optimal criteria to be developed for a variety of detection and estimation schemes. For example, one can develop optimal detection tests for the presence of cyclostationary behavior and estimate the cycle frequencies [9]. Finally, we note that the cross-spectrum and cross-correlation play a primary role in multi-sensor detection and estimation problems. For example, the time difference of arrival between sensors can be used to estimate the angle of arrival of a signal, a concept that easily extends to cyclostationary signals. Thus, the results of this report will prove useful in analysis of such multi-sensor cyclostationary signal processing scenarios.

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Appendix.—Proofs

Proof of Lemma 1

Using equation (12) in the left-hand side of equation (18) yields

$$\begin{aligned} & \left| s_{a_1, \dots, a_k}(t; \omega_1, \dots, \omega_k) - \sum_{\tau_1=-(T-1)}^{(T-1)} \dots \sum_{\tau_{k-1}=-(T-1)}^{(T-1)} c_{a_1, \dots, a_k}(t; \tau_1, \dots, \tau_{k-1}) e^{-j(\omega_1 \tau_1 + \dots + \omega_{k-1} \tau_{k-1})} \right| \\ & \leq \sum_{|\tau_1| > T} \dots \sum_{|\tau_{k-1}| > T} |c_{a_1, \dots, a_k}(t; \tau_1, \dots, \tau_{k-1})| \\ & \leq \sum_{|\tau_1| > T} \dots \sum_{|\tau_{k-1}| > T} \frac{|\tau_1| + \dots + |\tau_{k-1}|}{T} |c_{a_1, \dots, a_k}(t; \tau_1, \dots, \tau_{k-1})| = \mathcal{O}(T^{-1}), \end{aligned} \quad (42)$$

where the equality in the last line follows after applying Assumption 1a. \square

Proof of Corollary 1

Corollary 1 follows immediately from Lemma 1 with $k = 2$, $a_1 = i$, and $a_2 = j$. \square

Proof of Lemma 2

Using equation (5) and the multilinearity property of cumulants [3, eq (4.7)] we have

$$\begin{aligned} & \text{cum}\{X_{a_1}^{(T)}(\omega_1), \dots, X_{a_k}^{(T)}(\omega_k)\} \\ & = \sum_{t_1=0}^{T-1} \dots \sum_{t_{k-1}=0}^{T-1} \text{cum}\{x_{a_1}(t_1), \dots, x_{a_{k-1}}(t_{k-1})\} e^{-j(\omega_1 t_1 + \dots + \omega_k t_k)}. \end{aligned} \quad (43)$$

Using the assignments $t = t_k$, $\tau_1 = t_1 - t_k$, $\tau_2 = t_2 - t_k$, \dots , $\tau_{k-1} = t_{k-1} - t_k$, and $\lambda = \sum_{i=1}^k w_i$, we can rewrite equation (43) as

$$\begin{aligned} & \text{cum}\{X_{a_1}^{(T)}(\omega_1), \dots, X_{a_k}^{(T)}(\omega_k)\} \\ & = \sum_{t=0}^{T-1} \sum_{t+\tau_1=0}^{T-1} \dots \sum_{t+\tau_{k-1}=0}^{T-1} c_{a_1, \dots, a_k}(t; \tau_1, \dots, \tau_{k-1}) e^{-j\lambda t} e^{-j(\omega_1 \tau_1 + \dots + \omega_{k-1} \tau_{k-1})}, \end{aligned} \quad (44)$$

where we have made use of equations (8) and (10). Now, letting $t_a \triangleq -\min(\tau_1, \dots, \tau_{k-1}, 0)$, and $t_b \triangleq T-1 - \max(\tau_1, \dots, \tau_{k-1}, 0)$, then we may write,

$$\begin{aligned} & \text{cum}\{X_{a_1}^{(T)}(\omega_1), \dots, X_{a_k}^{(T)}(\omega_k)\} \\ & = \sum_{\tau_1=-(T-1)}^{T-1} \dots \sum_{\tau_{k-1}=-(T-1)}^{T-1} \sum_{t=t_a}^{t_b} e^{-j\lambda t} c_{a_1, \dots, a_k}(t; \tau_1, \dots, \tau_{k-1}) e^{-j(\omega_1 \tau_1 + \dots + \omega_{k-1} \tau_{k-1})}, \end{aligned} \quad (45)$$

for $0 \leq t_a \leq t_k \leq T-1$. If this last inequality is not satisfied then the limits of the summation are taken to be the empty set.

It may be shown that [3]

$$\sum_{t=t_a}^{t_k} e^{-j\lambda t} = \sum_{t=0}^{T-1} e^{-j\lambda t} R_\tau \quad (46)$$

where $R_\tau \leq |\gamma| [|\tau_1| + \dots + |\tau_{k-1}|]$, with γ a constant. Using this and the Fourier relation

$$c_{a_1, \dots, a_k}(t; \tau_1, \dots, \tau_{k-1}) = \sum_{\alpha} C_{a_1, \dots, a_k}(\alpha; \tau_1, \dots, \tau_{k-1}) e^{j\alpha t}, \quad (47)$$

allows us to write,

$$\begin{aligned} \text{cum}\{X_{a_1}^{(T)}(\omega_1), \dots, X_{a_k}^{(T)}(\omega_k)\} \\ = \sum_{t=0}^{T-1} e^{-j\lambda t} \sum_{\tau_1=-(T-1)}^{T-1} \dots \sum_{\tau_{k-1}=-(T-1)}^{T-1} e^{-j(\omega_1 \tau_1 + \dots + \omega_{k-1} \tau_{k-1})} c_{a_1, \dots, a_k}(t; \tau_1, \dots, \tau_{k-1}) \\ - R_\tau \sum_{\tau_1=-(T-1)}^{T-1} \dots \sum_{\tau_{k-1}=-(T-1)}^{T-1} \sum_{\alpha} C_{a_1, \dots, a_k}(\alpha; \tau_1, \dots, \tau_{k-1}). \end{aligned} \quad (48)$$

The second term on the right hand side of equation (48) is bounded, as we show next. First, use equation (47) with $t=0$ in the second term of equation (48), and then note that the result is less than or equal to

$$\sum_{\tau_1=-(T-1)}^{T-1} \dots \sum_{\tau_{k-1}=-(T-1)}^{T-1} [|\tau_1| + \dots + |\tau_{k-1}|] |c_{a_1, \dots, a_k}(0; \tau_1, \dots, \tau_{k-1})| \quad (49)$$

by applying Assumption 1a. Thus the second term of equation (48) is bounded as follows:

$$\begin{aligned} R_\tau \sum_{\tau_1=-(T-1)}^{T-1} \dots \sum_{\tau_{k-1}=-(T-1)}^{T-1} \sum_{\alpha} C_{a_1, \dots, a_k}(\alpha; \tau_1, \dots, \tau_{k-1}) \\ \leq \sum_{\tau_1=-(T-1)}^{T-1} \dots \sum_{\tau_{k-1}=-(T-1)}^{T-1} [|\tau_1| + \dots + |\tau_{k-1}|] c_{a_1, \dots, a_k}(0; \tau_1, \dots, \tau_{k-1}) = \mathcal{O}(1), \end{aligned} \quad (50)$$

and we can rewrite equation (48) as

$$\begin{aligned} \text{cum}\{X_{a_1}^{(T)}(\omega_1), \dots, X_{a_k}^{(T)}(\omega_k)\} \\ = \sum_{t=0}^{T-1} e^{-j\lambda t} \sum_{\tau_1=-(T-1)}^{T-1} \sum_{\tau_{k-1}=-(T-1)}^{T-1} e^{-j(\omega_1 \tau_1 + \dots + \omega_{k-1} \tau_{k-1})} c_{a_1, \dots, a_k}(t; \tau_1, \dots, \tau_{k-1}) + \mathcal{O}(1). \end{aligned} \quad (51)$$

The result follows by using Lemma 1 in equation (51). \square

Proof of Corollary 2

This result follows immediately from Lemma 2 with $k = 2$, $a_1 = i$, and $a_2 = j$. \square

Proof of Theorem 1

First we show that $I_{ij}^{(T)}(\alpha; \omega)$ is an asymptotically unbiased estimator of $S_{ij}(\alpha; \omega)$. Taking the expectation of the cyclic cross-periodogram yields

$$E[I_{ij}^{(T)}(\alpha; \omega)] = \frac{1}{T} \sum_{t=0}^{T-1} \sum_{s=0}^{T-1} E[x_i(t)x_j(t)]e^{-j\omega t}e^{-j(\alpha-\omega)s}. \quad (52)$$

Next we employ substitutions similar to those used in the proof of Lemma 2. Let $\tau = s - t$, $t_a = -\min(\tau, 0)$, and $t_b = T - 1 - \max(\tau, 0)$, leading to

$$E[I_{ij}^{(T)}(\alpha; \omega)] = \frac{1}{T} \sum_{t=t_a}^{t_b} e^{-j\alpha t} \sum_{\tau=-(T-1)}^{T-1} c_{ij}(t; \tau) e^{-j\omega \tau}. \quad (53)$$

Now, using Corollary 1, we obtain,

$$\begin{aligned} E[I_{ij}^{(T)}(\alpha; \omega)] &= \frac{1}{T} \sum_{t=t_a}^{t_b} e^{-j\alpha t} [s_{ij}(t; \omega) + \mathcal{O}(T^{-1})] \\ &= \frac{1}{T} \sum_{t=0}^T s_{ij}(t; \omega) e^{-j\alpha t} + o(1). \end{aligned} \quad (54)$$

That $I_{ij}^{(T)}(\alpha; \omega)$ is asymptotically unbiased is confirmed by taking the limit of equation (54) as $T \rightarrow \infty$, which results in $S_{ij}(\alpha; \omega)$.

Next we develop an expression for the asymptotic covariance of $I_{ij}^{(T)}(\alpha; \omega)$. A theorem of Brillinger [2, theorem 2.3.2], based on a previous result of Leonov and Shiryayev, provides a general method for expressing a joint cumulant as a sum of cumulants. Using this, we can express the covariance of the cyclic cross periodogram as,

$$\begin{aligned} \text{cov}\{I_{ij}^{(T)}(\alpha; \omega), I_{kl}^{(T)}(\beta; \mu)\} &= \text{cum}\{X_i^{(T)}(\omega)X_j^{(T)}(\alpha - \omega), X_k^{(T)}(-\mu)X_l^{(T)}(\mu - \beta)\} \\ &= \frac{1}{T^2} \text{cum}\{X_i^{(T)}(\omega), X_j^{(T)}(\alpha - \omega), X_k^{(T)}(-\mu), X_l^{(T)}(\mu - \beta)\} \\ &+ \frac{1}{T^2} \text{cum}\{X_i^{(T)}(\omega), X_k^{(T)}(-\mu)\} \text{cum}\{X_j^{(T)}(\alpha - \omega), X_l^{(T)}(\mu - \beta)\} \\ &+ \frac{1}{T^2} \text{cum}\{X_i^{(T)}(\omega), X_l^{(T)}(\mu - \beta)\} \text{cum}\{X_j^{(T)}(\alpha - \omega), X_k^{(T)}(-\mu)\}, \end{aligned} \quad (55)$$

where we have made use of the fact that $\text{cov}\{x, y\} = \text{cum}\{x, y^*\}$ [2].
Using Lemma 2 and Corollary 2 in equation (55) yields

$$\begin{aligned}
& \text{cov}\{I_{ij}^{(T)}(\alpha; \omega), I_{kl}^{(T)}(\beta; \mu)\} \\
&= \frac{1}{T^2} \sum_{t=0}^{T-1} S_{ijkl}(t; \omega, \alpha - \omega, -\mu) e^{-j(\alpha - \beta)t} \\
&+ \left[\frac{1}{T} \sum_{t=0}^{T-1} S_{ik}(t; \omega) e^{-j(\omega - \mu)t} \right] \left[\frac{1}{T} \sum_{t=0}^{T-1} S_{jl}(t; \alpha - \omega) e^{-j(\alpha - \omega + \mu - \beta)t} \right] \\
&+ \left[\frac{1}{T} \sum_{t=0}^{T-1} S_{il}(t; \omega) e^{-j(\omega + \mu - \beta)t} \right] \left[\frac{1}{T} \sum_{t=0}^{T-1} S_{jk}(t; \alpha - \omega) e^{-j(\alpha - \omega - \mu)t} \right] + \mathcal{O}(T^{-1}).
\end{aligned} \tag{56}$$

By assumption, $S_{ijkl}(\alpha; \omega_1, \omega_2, \omega_3)$ exists and is finite. Therefore, the first term of equation (56) goes to zero as $T \rightarrow \infty$, and we are left with

$$\begin{aligned}
\lim_{T \rightarrow \infty} \text{cov}\{I_{ij}^{(T)}(\alpha; \omega), I_{kl}^{(T)}(\beta; \mu)\} &= S_{ik}(\omega - \mu; \omega) S_{jl}(\alpha - \omega + \mu - \beta; \alpha - \omega) \\
&+ S_{il}(\omega + \mu - \beta; \omega) S_{jk}(\alpha - \omega - \mu; \alpha - \omega). \quad \square
\end{aligned} \tag{57}$$

Proof of Theorem 2

The proof of Theorem 2 is somewhat lengthy, but it follows along the lines of Dandawate's Theorems 2.3.3 and 2.3.5. For this reason we omit the detailed proof here. The proof of equations (40) and (41) follows by generalizing Dandawate's Theorem 2.3.3 to the cross-periodogram case. Proof of the asymptotic normality can be achieved by showing that the higher order ($k > 2$) cumulants of the estimate vanish asymptotically, as in Dandawate's Theorem 2.3.5. \square

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